Lecture 1

Risk Aware and Robust Nonlinear Planning

Introduction and Course Overview

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Topics:

- Introduction to Planning Under Uncertainty
- Approaches and Challenges
- Technical Idea and Mathematical Tools
- Applications
Introduction to Planning Under Uncertainty
Planning For Autonomous Systems
Challenge: Uncertainty

- **Planning under Uncertainty**: Planning in presence of imperfect or unknown information.

- Due to uncertainty, it is impossible to exactly describe the “current situation” or “future behavior” of the systems/environment.
Challenge: **Uncertainty**

- **Planning under Uncertainty**: Planning in presence of imperfect or unknown information.

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**Goal State**

- Planned trajectory from initial state to the goal state.
- Actual trajectory due to uncertainties.
Source of Uncertainty

1. Environment:
   i) Sensor Noise
e.g., localizing obstacles or the robot
   ii) Control Disturbance
e.g., wind disturbances
   iii) Unmodeled Environment
e.g., rough train
   iii) Intention
e.g., future behavior of other agents (dynamic environment)

2. System:
   i) Imperfect system model
e.g., unknow parameters of system model
      unmodeled dynamics (linear model for nonlinear systems)
How to Deal with Uncertainty?
How to Deal with Uncertainty?

- Robust Approaches
- Risk Bounded Approaches
Robust Approaches:

- Plan should be valid for all possible realization of uncertainty
- Look at the Uncertainty Set (Range of Uncertainty)
Motion Planning

Goal

Obstacle (turtlebot)

Start
Motion Planning
Motion Planning Under Uncertainty:
Motion Planning Under Uncertainty:
Motion Planning Under Uncertainty:

Start

Moving Obstacle

Goal
Motion Planning Under Uncertainty:

There is no path from start to goal region that is valid for all possible realization of uncertainty.

Robust approach

Conservative solution

Goal

Moving Obstacle

Start
Motion Planning Under Uncertainty:

There is no path from start to goal region that is valid for all possible realization of uncertainty.

Robust approach

Conservative solution

Takes 10 days
Motion Planning Under Uncertainty:

Risk Bounded Approach:
- Look at “frequency of realization” of Uncertainty

Goal

Start
Motion Planning Under Uncertainty:

Risk Bounded Approach:
- Look at “frequency of realization” of Uncertainty
**Motion Planning Under Uncertainty:**

**Risk Bounded Approach:**
- Look at “frequency of realization” of Uncertainty

Probability of Collision $\leq \Delta \approx 0$
Risk Bounded Approaches

- Plan should be valid with high probability.
- Look at “frequency of realization” of Uncertainty (Probability Distribution)
Risk Bounded VS Robust

Probability VS Possibility
Risk Bounded

0 ↔ 1
Topics:

- Introduction to Planning Under Uncertainty

- Approaches and Challenges

- Technical Idea and Mathematical Tools

- Applications
Optimization Based Planning
Optimization Based Planning

\[
\begin{align*}
\text{minimize} & \quad \text{Objective Function (design parameters)} \\
\text{subject to} & \quad \text{Constraints (design parameters)}
\end{align*}
\]

**Objective function:** cost of execution

**Constraints:** safety constraints, resource constraints, dynamical constraints, temporal constraints
Example: Trajectory Planning

Dynamical Systems

Continuous State space model

\[ x_{k+1} = f(x_k, u_k) \]

- **\( x \):** joint angles and angular velocities
- **\( u \):** Torque motor

- **\( u \):** Steering angle, Torque

- **\( x \):** position and velocity

\[ x_1, u_1 \]

\[ x_2, u_2 \]

\[ x_3, u_3 \]

\[ x_4, u_4 \]

\[ x_5 \]

\[ u_1 \]

\[ u_2 \]
Example: Trajectory Planning

Find a sequence of control inputs $[u_0, \ldots, u_{N-1}]$ to derive the robot to the goal point.

\[
\begin{align*}
\text{minimize} & \quad \sum_{k=0}^{N-1} u^2(k) \\
\text{subject to} & \quad x_0 = x_0^*, x_N = x_N^* \\
& \quad x_{k+1} = f(x_k, u_k) \\
& \quad x_k \in \chi_{safety} \\
& \quad u_k \in U
\end{align*}
\]

- Control cost
- Boundary conditions
- Dynamical constraints
- Safety constraints
- Resource constraints
Optimization Based Planning

minimize \quad \text{Objective Function (design parameters)}

\text{design parameters}

subject to \quad \text{Constraints (design parameters)}

\text{Mathematical Formulation}

minimize \quad p(x)

_{x \in \mathbb{R}^n}

subject to \quad g_i(x) \geq 0, \quad i = 1, \ldots, n_g

p(x) : \mathbb{R}^n \to \mathbb{R}, \quad g_i(x) : \mathbb{R}^n \to \mathbb{R} \quad i = 1, \ldots, n_g
Optimization Based Planning Under Uncertainty

- Robust Optimization
- Risk Aware Optimization, i.e., Chance optimization and Chance Constrained Optimization
- Distributionally Robust Optimization
Robust Optimization Based Planning

minimize \[ p(x) \] \[ \text{subject to} \quad g_i(x, \omega) \geq 0, \quad i = 1, \ldots, n_g, \quad \forall \omega \in \Omega \]

Objective Function (design parameters)
satisfy constraints for all possible value of uncertainties
Example: Robust Trajectory Optimization

Uncertain Dynamical Systems and Uncertain Safety Constraints

Continuous State space model

\[ x_{k+1} = f(x_k, u_k, \omega_k) \]

- **states**
- **inputs**
- **Uncertainty, \( \omega_k \in \Omega \)**

Uncertain Obstacle

\[ g(x, \omega_{obs}) \geq 0 \]

- **Obstacle(\( \omega_{obs} \))**
- Uncertainty parameters of the obstacle \( \omega_{obs} \in \Omega_{obs} \)

Example: Uncertain circular obstacle

\[ \omega_{obs}^2 - x_1^2 - x_2^2 \geq 0 \]

\( \omega_{obs} \in [\omega_{obs_1}, \omega_{obs_2}] \)
Example: Robust Trajectory Optimization

Find a sequence of control inputs \([u_0, \ldots, u_{N-1}]\) to derive the robot to the goal region in the presence of uncertainties.

\[
\begin{align*}
\text{minimize} & \quad \sum_{k=0}^{N-1} u^2(k) \\
\text{subject to} & \quad x_0 = x_0^*, x_N \in x_N^* \\
& \quad x_{k+1} = f(x_k, u_k, \omega_k) \\
& \quad x_k \in \chi_{safe}(\omega_{obs}) \\
& \quad \forall \omega_{x_k} \in \Omega_x, \forall \omega_{obs} \in \Omega_{obs} \\
& \quad u_k \in \mathcal{U}
\end{align*}
\]
Robust Optimization Based Planning

Minimize \( p(x) \) subject to \( g_i(x, \omega) \geq 0, \ i = 1, \ldots, n_g, \ \forall \omega \in \Omega \)

Objective Function (design parameters)

Minimize design parameters subject to satisfy constraints for all possible vaulese of uncertainties

Mathematical Formulation

Uncertainty

Uncertainty Set
3. Risk Aware Optimization Based Planning

**Chance Optimization**

\[
\text{maximize} \quad \text{Probability}(\text{Success}(\text{design parameters, probabilistic uncertainty}))
\]

subject to \quad \text{Constraints}(\text{design parameters})

**Chance Constrained Optimization**

\[
\text{minimize} \quad \text{Objective Function}(\text{design parameters})
\]

subject to \quad \text{Probability}(\text{Success}(\text{design parameters, probabilistic uncertainty})) \geq 1 - \Delta

\text{Acceptable risk level}
Example: Chance Constrained Trajectory Optimization

Probabilistic Dynamical Systems and Probabilistic Safety Constraints

Continuous State space model:

\[ x_{k+1} = f(x_k, u_k, \omega_k) \]

\( \omega_k \sim \text{pr}(\omega_k) \)

Family of trajectories Due to uncertainty

Probabilistic Obstacle:

\[ p(x, \omega_{obs}) \geq 0 \]

\( \omega_{obs} \sim \text{pr}(\omega_{obs}) \)

Example: probabilistic circular obstacle

\[ \omega_{obs}^2 - x_1^2 - x_2^2 \geq 0 \]

\( \omega_{obs} \sim \text{pr}(\omega_{obs}) \)
Example: Chance Constrained Trajectory Optimization

Find a sequence of control inputs $[u_0, \ldots, u_{N-1}]$ to derive the robot to the goal region in the presence of probabilistic uncertainties.

\[
\begin{align*}
\text{minimize} & \quad \sum_{k=0}^{N-1} u^2(k) \\
\text{subject to} & \quad \mathbb{E}[x_N] \in x_N^* \\
& \quad x_{k+1} = f(x_k, u_k, \omega_k) \\
& \quad \text{Prob}(x_k \in \chi_{safe}(\omega_{obs})) \geq 1 - \Delta \\
& \quad x_0 \sim \text{pr}(x), \quad \omega_k \sim \text{pr}(\omega_k) \\
& \quad u_k \in \mathcal{U}
\end{align*}
\]

- **Success** = Remaining Safe
Example: Chance Trajectory Optimization

Find a sequence of control inputs $[u_0, \ldots, u_{N-1}]$ to derive the robot to the goal region in the presence of probabilistic uncertainties.

- **Success** = Remaining safe and reaching the goal

$$\max_{u_k | k=0}^{N-1} \quad \text{Prob}(x_k \in \chi_{safe}(\omega_{obs}), x_N \in x^*_N)$$

subject to

$x_{k+1} = f(x_k, u_k, \omega_k)$

$u_k \in U$

$\chi_{safe}(\omega_{obs})$

$\chi^*_N$

$\omega_{obs}$

$x_k(\omega_k)$

Obstacle

Family of trajectories

Due to uncertainty

Remaining safe and reaching the goal
Example: Portfolio Selection Problem

- Assets with uncertain rate of return $\omega_i \sim p_r(\omega)$, $i = 1, ..., 4$
- $x_i$ invested money in asset $i$

- **Success** = Achieve a return higher than "$r^*$"
  $\{\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 x_4 \geq r^*\}$

### Chance Optimization

$$\begin{align*}
\text{maximize} & \quad \text{Probability}(\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 x_4 \geq r^*) \\
\text{subject to} & \quad x_1 + x_2 + x_3 + x_4 \leq \chi 
\end{align*}$$

### Chance Constrained Optimization

$$\begin{align*}
\text{minimize} & \quad x_1 + x_2 + x_3 + x_4 \\
\text{subject to} & \quad \text{Probability}(\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 x_4 \geq r^*) \geq 1 - \Delta
\end{align*}$$
3. Risk Aware Optimization Based Planning

Mathematical Formulation

**Chance Optimization**

\[
\begin{align*}
\text{maximize} & \quad \frac{\text{Probability}_{\Pr(\omega)}( p_i(x, \omega) \geq 0, \ i = 1, \ldots, n_p )}{x \in \mathbb{R}^n} \\
\text{subject to} & \quad g_i(x) \geq 0, \ i = 1, \ldots, n_g
\end{align*}
\]

**Chance Constrained Optimization**

\[
\begin{align*}
\text{minimize} & \quad p(x) \\
\text{subject to} & \quad \text{Probability}_{\Pr(\omega)}( g_i(x, \omega) \geq 0, \ i = 1, \ldots, n_g ) \geq 1 - \Delta
\end{align*}
\]
4. Distributionally Robust Chance Constraint Optimization

- Probability distribution with uncertain parameters $a \in \mathcal{A}$
  
  e.g., Gaussian probability distribution with uncertain mean and deviation
  
  $$
  \frac{1}{\sqrt{2\pi a_2}} e^{-\frac{(x-a_1)^2}{2a_2^2}}, \quad a_1 \in [l_1, u_2], a_2 \in [l_2, u_2]
  $$

  Family of probability distributions

\[\text{minimize} \quad \text{Objective Function(design parameters)}\]
\[\text{subject to} \quad \text{uncertainty} \sim \text{Probability distribution}(a), \ a \in \mathcal{A}\]
\[\text{Probability(Success( design parameters, probabilistic uncertainty))) } \geq 1 - \Delta, \ \forall a \in \mathcal{A}\]

Chance constraints should be satisfied for the family of the probability distributions of the uncertainties
4. Distributionally Robust Chance Constrained Optimization

**Mathematical Formulation**

```
minimize \frac{\text{Objective Function}}{\text{design parameters}}

subject to 
\begin{align*}
&\text{uncertainty } \sim \text{ Probability distribution}(a), \ a \in \mathcal{A} \\
&\text{Probability} \left( \text{Success( design parameters, probabilistic uncertainty))} \right) \geq 1 - \Delta, \ \forall a \in \mathcal{A} \\
\end{align*}

Chance constraints should be satisfied for the family of the probability distributions of the uncertainties
```

```
minimize \frac{p(x)}{x \in \mathbb{R}^n}

subject to 
\begin{align*}
&\omega \sim \text{pr(}\omega, a\text{)}, \ a \in \mathcal{A} \\
&\text{Probability}_{\text{pr}(\omega,a)}( \ g_i(x, \omega) \geq 0, \ i = 1, \ldots, n_g \ ) \geq 1 - \Delta, \ \forall a \in \mathcal{A} \\
\end{align*}
```
<table>
<thead>
<tr>
<th>Optimization Type</th>
<th>Mathematical Formulation</th>
</tr>
</thead>
</table>
| **Nonlinear Optimization**        | \[
\begin{align*}
\text{minimize} & \quad p(x) \\
\text{subject to} & \quad g_i(x) \geq 0, \quad i = 1, \ldots, n_g
\end{align*}
\] |
| **Robust Optimization**           | \[
\begin{align*}
\text{minimize} & \quad p(x) \\
\text{subject to} & \quad g_i(x, \omega) \geq 0, \quad i = 1, \ldots, n_g, \quad \forall \omega \in \Omega
\end{align*}
\] |
| **Chance Optimization**           | \[
\begin{align*}
\text{maximize} & \quad \text{Probability}_{pr(\omega)}( p_i(x, \omega) \geq 0, \quad i = 1, \ldots, n_p ) \\
\text{subject to} & \quad g_i(x) \geq 0, \quad i = 1, \ldots, n_g
\end{align*}
\] |
| **Chance Constrained Optimization** | \[
\begin{align*}
\text{minimize} & \quad p(x) \\
\text{subject to} & \quad \text{Probability}_{pr(\omega)}( g_i(x, \omega) \geq 0, \quad i = 1, \ldots, n_g ) \geq 1 - \Delta
\end{align*}
\] |
| **Distributionally Robust Chance Constrained Optimization** | \[
\begin{align*}
\text{minimize} & \quad p(x) \\
\text{subject to} & \quad \omega \sim \text{pr}(\omega, a), \quad a \in \mathcal{A} \\
& \quad \text{Probability}_{\text{pr}(\omega, a)}( g_i(x, \omega) \geq 0, \quad i = 1, \ldots, n_g ) \geq 1 - \Delta, \quad \forall a \in \mathcal{A}
\end{align*}
\] |
Purpose of this course:

- State-of-the-art techniques to efficiently solve nonlinear, robust, and risk aware optimization problems.

- Application in analyze and control of uncertain nonlinear dynamical systems.
Assumption

Objective function and constraints of optimization problems, \( p, g_i \), are polynomial functions.

- **Polynomial in “\( x \)” is a finite linear combination of powers of “\( x \)”**

\[
p(x_1) = 1 + 0.5x_1^2 + 0.75x_1^3
\]

Polynomial of degree 3

\[
p(x_1, x_2) = 0.56 + 0.5x_1 + 2x_2^2 + 0.75x_1^3x_2^2
\]

Polynomial of degree 5

**Stone-Weierstrass Theorem:** Every **continuous function** defined on a closed set can be uniformly approximated as closely as desired by a polynomial function.
Example

- Polynomial dynamical system
  \[
  \begin{align*}
  x_1(k+1) &= x_1(k) + 0.2(x_1(k) - \frac{x_1^3(k)}{3}) - x_2(k) + 0.875 \\
  x_2(k+1) &= x_2(k) + 0.016(x_1(k) - 0.8x_2 + 0.7)
  \end{align*}
  \]

- Polynomial constraint

\[
0.42x_1^5 - 1.2x_1^4x_2 - 0.48x_1^4 + 0.3x_1^3x_2^2 - 0.57x_1^3x_2 + 0.61x_1^3 - 0.66x_1^2x_2^3 + 0.17x_1^2x_2^2 + 1.9x_1^2x_2 + 0.066x_1^2 + 0.69x_1x_2^4 - 0.14x_1x_2^3 - 0.85x_1x_2^2 + 0.6x_1x_2 - 0.22x_1 + 0.011x_1^5 - 0.068x_2^4 - 0.07x_2^3 - 0.42x_2^2 - 0.084x_2 + 0.84 \geq 0.8
\]
Optimization Based Planning Under Uncertainty

- Challenges
1. Challenge: Nonconvexities

- Nonlinear, robust, and risk aware optimization problems are in general nonconvex problems.
Nonconvex Optimization

- Multiple local minima
- Sensitive to initial point

Convex Optimization

- Unique minimum: global/local
2. Challenge: Chance and Robust Constraints Evaluation

**Chance Constraint**

\[ \text{Probability}_{pr(\omega)}( g(x, \omega) \geq 0 ) \geq 1 - \Delta \]

**Robust Constraint**

\[ g(x, \omega) \geq 0 \quad \forall \omega \in \Omega \]
Chance Constraint Evaluation

**Chance Constraint:**

\[
\text{Probability}_{pr(\omega)} ( g(x, \omega) \geq 0 ) = \int_{g(x,\omega) \geq 0} \text{pr}(\omega)d\omega
\]

- Multivariate integral
- In general, it does not have any analytical solution

- Sampling based methods (e.g., Monte-Carlo methods) DO NOT provide any guarantee.

Probability $\geq 1 - \Delta$ \[\text{Replace} \] Estimation of Probability $\geq 1 - \Delta$
Robust Constraint Evaluation

Robust Constraint \[ g(x, \omega) \geq 0 \quad \forall \omega \in \Omega \]

- This results in Infinite number of constraints \( g(x, \omega_i) \geq 0, \omega_i \in \Omega \)
3. Challenge: Uncertainty Propagation

Continuous State space model

\[ x_{k+1} = f(x_k, u_k, \omega_k) \]

uncertainties
\[ x_0 \sim \text{pr}(x), \]
\[ \omega_k \sim \text{pr}(\omega) \]
Uncertainty propagation

Example:

\[ x_{k+1} = x_k + v_k \cos(\theta_k) \]
\[ y_{k+1} = y_k + v_k \sin(\theta_k) \]

states: \((x, y)\) position

control inputs: \((\theta, v)\) yaw angle and velocity

Planned Control Inputs:

![Graphs showing planned control inputs for yaw angle and velocity over time.](image)
\[ x_{k+1} = x_k + (v_k + \omega_{1k}) \cos(\theta_k + \omega_{2k}) + \omega_{3k} \]
\[ y_{k+1} = y_k + (v_k + \omega_{1k}) \sin(\theta_k + \omega_{2k}) + \omega_{4k} \]

**states:** \((x, y)\) position  
**control inputs:** \((\theta, v)\) yaw angle and velocity  
**uncertainty:** \((\omega_1, \omega_2, \omega_3, \omega_4)\)
\[
\begin{align*}
x_{k+1} &= x_k + (v_k + \omega_{1k}) \cos(\theta_k + \omega_{2k}) + \omega_{3k} \\
y_{k+1} &= y_k + (v_k + \omega_{1k}) \sin(\theta_k + \omega_{2k}) + \omega_{4k}
\end{align*}
\]

Probability distributions:

\[
\begin{align*}
\omega_1 &\sim \text{Uniform}[-0.1, 0.1] \\
\omega_2 &\sim \text{Uniform}[-1, 1] \\
\omega_3, \omega_4 &\sim \text{Beta}[-0.1, 0.1]
\end{align*}
\]
Probability distributions of states of the system

\[ x_k \sim p(x_k) \]
For safety verification, we need to obtain probability distributions of the states of the system.

For this, we need to propagate initial probability distribution of the states through nonlinear dynamics of the system.
Optimization Based Planning

- Challenges:

  1) Nonconvexities
  2) Evaluation of Chance constraint and Robust Constraints
  3) Uncertainty Propagation Through Nonlinear Systems
Topics:

- Introduction to Planning Under Uncertainty
- Approaches and Challenges
  - Technical Idea and Mathematical Tools
- Applications
Optimization Based Planning Under Uncertainty

- Main Idea
Main Idea: Convexification

- To efficiently solve the nonlinear, robust, and risk aware optimization problems, we look for **convex relaxation** of the optimization problems.

- Convex optimization in form of Semidefinite Program (SDP).

Nonconvex Optimization
Multiple local optima

Convex Optimization
Unique optimum: global/local

nonlinear, robust, and risk aware optimization problems

Semidefinite Program
Convex Optimization

Linear Program:

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

Example

\[
\begin{align*}
\text{min}_x & \quad 3x_1 + 5x_2 + x_3 \\
\text{s.t.} & \quad x_1 + 3x_2 + 5x_3 = 2 \\
& \quad x_1 + 9x_2 + 4x_3 = 1 \\
& \quad x_1 \geq 0, \ x_2 \geq 0
\end{align*}
\]
Convex Optimization

Linear Program:

\[
\begin{align*}
& \text{minimize} & & c^T x \\
& \text{subject to} & & \begin{cases} 
Ax = b \\
x \geq 0
\end{cases}
\end{align*}
\]

Example

Find \([x_1, x_2, x_3]\) to

\[
\min_x 3x_1 + 5x_2 + x_3 \\
\begin{align*}
x_1 + 3x_2 + 5x_3 &= 2 \\
x_1 + 9x_2 + 4x_3 &= 1 \\
x_1 \geq 0, \ x_2 \geq 0
\end{align*}
\]

Semidefinite Program:

\[
\begin{align*}
& \text{minimize} & & \text{linear function} \\
& \text{subject to} & & \begin{cases} 
C \cdot X \text{---linear function} \\
A \cdot X = b \text{---linear constraints} \\
X \succeq 0 \text{---linear matrix inequalities}
\end{cases}
\end{align*}
\]

Example

\[
X = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix}
\]

\[
\begin{align*}
& \min_x 3x_{11} + 5x_{12} + x_{22} \\
& \begin{align*}
x_{11} + 3x_{12} + 5x_{22} &= 2 \\
x_{11} + 9x_{12} + 4x_{22} &= 1 \\
x_1 \geq 0
\end{align*}
\end{align*}
\]
Main Idea: Convexification

- To efficiently solve the nonlinear, robust, and risk aware optimization problems, we look for convex relaxation of the optimization problems.

- Convex optimization in form of Semidefinite Program (SDP).
Optimization Based Planning Under Uncertainty

- Mathematical Tools
nonlinear, robust, and risk aware optimization problems

Convexification:

Tools:
  i) Theory of Nonnegative Polynomials
  ii) Theory of Moments

Semidefinite Program
Nonnegative Polynomial Based SDP relaxation

Nonnegative Polynomials

\[ P(x) \geq 0 \quad \forall x \in \mathbb{R}^n \]
Nonnegative Polynomial Based SDP relaxation

Nonlinear Optimization

\[
\begin{align*}
\text{minimize} & \quad p(x) \\
\text{subject to} & \quad g_i(x) \geq 0, \quad i = 1, \ldots, n_g
\end{align*}
\]

SOS based SDP Relaxation

SDP in terms of coefficients of \( P(x_1, x_2, \ldots, x_n) \geq 0 \)

Main Idea:
Instead of looking for decision parameters \((x_1, x_2, \ldots, x_n)\), we look for a nonnegative polynomial in terms of decision parameters, i.e. \( P(x_1, x_2, \ldots, x_n) \geq 0 \)
Nonnegative Polynomial Based SDP relaxation

Nonlinear Optimization
\[
\begin{align*}
\text{minimize} \quad & p(x) \\
\text{subject to} \quad & g_i(x) \geq 0, \ i = 1, \ldots, n_g
\end{align*}
\]

SOS based SDP Relaxation
SDP in terms of coefficients of \( P(x_1, x_2, \ldots, x_n) \geq 0 \)

Main Idea:
Instead of looking for decision parameters \((x_1, x_2, \ldots, x_n)\), we look for a nonnegative polynomial in terms of decision parameters, i.e. \( P(x_1, x_2, \ldots, x_n) \geq 0 \)

- We translate objective function and constraints of the original optimization problem in terms of coefficients of nonnegative polynomial \( P(x_1, x_2, \ldots, x_n) \).
Nonnegative Polynomial Based SDP relaxation

| Nonlinear Optimization | \[ \min_{x \in \mathbb{R}^n} \quad p(x) \]
|                        | subject to \[ g_i(x) \geq 0, \quad i = 1, \ldots, n_g \] |

| SOS based SDP Relaxation | SDP in terms of coefficients of \( P(x_1, x_2, \ldots, x_n) \geq 0 \) |

Main Idea:
Instead of looking for decision parameters \((x_1, x_2, \ldots, x_n)\), we look for a nonnegative polynomial in terms of decision parameters, i.e. \( P(x_1, x_2, \ldots, x_n) \geq 0 \)

- We translate objective function and constraints of the original optimization problem in terms of coefficients of nonnegative polynomial \( P(x_1, x_2, \ldots, x_n) \).
- We use nonnegativity condition for polynomial \( P(x_1, x_2, \ldots, x_n) \). (i.e., sum of squares (SOS) condition)
Nonnegative Polynomial Based SDP relaxation

Nonlinear Optimization

\[
\begin{align*}
\text{minimize} & \quad p(x) \\
\text{subject to} & \quad g_i(x) \geq 0, \quad i = 1, \ldots, n_g
\end{align*}
\]

SOS based SDP Relaxation

SDP in terms of coefficients of \( P(x_1, x_2, \ldots, x_n) \geq 0 \)

Main Idea:
Instead of looking for decision parameters \((x_1, x_2, \ldots, x_n)\), we look for a nonnegative polynomial in terms of decision parameters, i.e. \( P(x_1, x_2, \ldots, x_n) \geq 0 \)

- We translate objective function and constraints of the original optimization problem in terms of coefficients of nonnegative polynomial \( P(x_1, x_2, \ldots, x_n) \).
- We use nonnegativity condition for polynomial \( P(x_1, x_2, \ldots, x_n) \). (i.e., sum of squares (SOS)condition)
- This results in an SDP in terms of coefficients of \( P(x_1, x_2, \ldots, x_n) \). (SOS based SDP)
**Main Idea:**

- We treat decision variables \((x_1, x_2, \ldots, x_n)\) as random variable.

- Instead of looking for decision parameters \((x_1, x_2, \ldots, x_n)\), we look for its probability distribution, i.e. \(\text{pr}(x_1, x_2, \ldots, x_n)\).

- Later, we extract the deterministic solution \((x_1, x_2, \ldots, x_n)\).
Nonlinear Optimization

\[
\begin{align*}
\text{minimize} & \quad p(x) \\
\text{subject to} & \quad g_i(x) \geq 0, \quad i = 1, \ldots, n_g
\end{align*}
\]

Main Idea:

• We treat decision variables \((x_1, x_2, \ldots, x_n)\) as random variable.

• Instead of looking for decision parameters \((x_1, x_2, \ldots, x_n)\), we look for its probability distribution, i.e. \(\text{pr}(x_1, x_2, \ldots, x_n)\)

• Later, we extract the deterministic solution \((x_1, x_2, \ldots, x_n)\).

To obtain an SDP formulation, instead of looking for probability distribution \(\text{pr}(x_1, x_2, \ldots, x_n)\), we look for its statistics called \text{moments}. 
Moments of probability distributions

**moment of order** $\alpha$

$$y_\alpha = E[x^\alpha] = \int x^\alpha p(x)dx = \int x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}pr(x)dx$$

- 1-st moment (mean): $y_1 = E[x^1] = \int xpr(x)dx$
- 2-nd moment: $y_2 = E[x^2] = \int x^2pr(x)dx$
Moments of probability distributions

**moment of order** $\alpha$

$$y_\alpha = E[x^\alpha] = \int x^\alpha pr(x) \, dx = \int x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} pr(x) \, dx$$

- 1-st moment *(mean)*:  
  $$y_1 = E[x^1] = \int x pr(x) \, dx$$

- 2-nd moment:  
  $$y_2 = E[x^2] = \int x^2 pr(x) \, dx$$

  $$\text{var} = E[(x - E[x])^2] = y_2 - y_1^2$$
Moment Based SDP Relaxation

**Nonlinear Optimization**

$$\begin{align*}
\text{minimize} & \quad p(x) \\
\text{subject to} & \quad g_i(x) \geq 0, \quad i = 1, \ldots, n_g
\end{align*}$$

**Moment based SDP Relaxation**

SDP in terms of moments of $\text{pr}(x_1, x_2, \ldots, x_n)$
Moment Based SDP Relaxation

Nonlinear Optimization

\[
\begin{align*}
\text{minimize} & \quad p(x) \\
\text{subject to} & \quad g_i(x) \geq 0, \quad i = 1, \ldots, n_g
\end{align*}
\]

Moment based SDP Relaxation

SDP in terms of moments of \( pr(x_1, x_2, \ldots, x_n) \)

- We translate objective function and constraints of the original optimization problem in terms of the moments of probability distribution \( pr(x_1, x_2, \ldots, x_n) \).

- This results in an SDP in terms of the moment. (*Moment based SDP*)
We will apply these techniques to the Uncertain optimization problems
### Nonlinear Optimization

$$\begin{align*}
\text{minimize} & \quad p(x) \\
\text{subject to} & \quad g_i(x) \geq 0, \quad i = 1, \ldots, n_g
\end{align*}$$

### Robust Optimization

$$\begin{align*}
\text{minimize} & \quad p(x) \\
\text{subject to} & \quad g_i(x, \omega) \geq 0, \quad i = 1, \ldots, n_g, \quad \forall \omega \in \Omega
\end{align*}$$

### Chance Optimization

$$\begin{align*}
\text{maximize} & \quad \text{Probability}_{\text{pr}(\omega)}( p_i(x, \omega) \geq 0, \quad i = 1, \ldots, n_p ) \\
\text{subject to} & \quad g_i(x) \geq 0, \quad i = 1, \ldots, n_g
\end{align*}$$

### Chance Constrained Optimization

$$\begin{align*}
\text{minimize} & \quad p(x) \\
\text{subject to} & \quad \text{Probability}_{\text{pr}(\omega)}( g_i(x, \omega) \geq 0, \quad i = 1, \ldots, n_g ) \geq 1 - \Delta
\end{align*}$$

### Distributionally Robust Chance Constrained Optimization

$$\begin{align*}
\text{minimize} & \quad p(x) \\
\text{subject to} & \quad \omega \sim \text{pr}(\omega, a), \quad a \in \mathcal{A} \\
& \quad \text{Probability}_{\text{pr}(\omega, a)}( g_i(x, \omega) \geq 0, \quad i = 1, \ldots, n_g ) \geq 1 - \Delta, \quad \forall a \in \mathcal{A}
\end{align*}$$
1) Optimization Based Planning under Uncertainty
   i) Nonlinear Optimization
   ii) Robust Optimization
   iii) Chance Optimization/Chance Constrained Optimization
   iv) Distributionally Robust Chance Constrained Optimization

2) Challenges
   i) Nonconvexities
   ii) Evaluation of Chance constraint and Robust Constraints
   iii) Uncertainty Propagation Through Nonlinear Systems

3) Main Idea:
   Replace the nonconvex optimization with Convex optimization in the form of
   Semidefinite Program (SDP).

4) We solve Moment/SOS based SDP
Topics:

- Introduction to Planning Under Uncertainty
- Approaches and Challenges
- Technical Idea and Mathematical Tools
- Applications
Optimization Based Planning Under Uncertainty

- Applications
Optimization Based Planning Under Uncertainty

1. Safety Verification for Probabilistic Systems
2. Risk Aware Control and Planning
3. Dynamical system with Gaussian Uncertainties
4. Occupation Measure Based Control and Analyze
5. Sum-of-Squares Based Robust Control and Analyze of Uncertain Systems
1. Safety Verification
1.1 Risk Estimation

- Probabilistic location of the robot
- Nonconvex obstacle with probabilistic uncertainties (uncertain size, location, geometry)

- **Risk**: probability of collision with obstacle

**Find**: Lower/Upper bounds of the risk

\[ P_{\text{risk}}^L \leq P_{\text{risk}}^* \leq P_{\text{risk}}^U \]

- Particular case of “chance optimization”
- SOS/Moment based SDP formulation
1.2 Risk Estimation and Uncertainty Propagation

- Initial Probabilistic location of the robot
- Candidate plan
e.g., nominal path and control input \((x^*_k, u^*_k)\) \(k = [0, \ldots, N - 1]\)

➢ **Risk**: probability of collision with obstacle

Find: Lower/Upper bounds of the risk at time \(k = [0, \ldots, N - 1]\) for given \((x^*_k, u^*_k)\)

- We need to find \(x_k \sim pr(x_k)\). We find moment sequence of \(pr(x_k)\) using the uncertain nonlinear dynamics
- Solve Risk estimation problem at each time \(k\)
1.3 Uncertainty Set Construction

- Initial Probabilistic location of the robot
- Candidate plan
e.g., nominal path and control input \((x_k^*, u_k^*)\) \(k = [0, \ldots N - 1]\)

- Nonconvex obstacle with probabilistic uncertainties
  (uncertain size, location, geometry)

- **Risk**: probability of collision with obstacle

- Instead of looking for \(x_k \sim \text{pr}(x_k)\), we find the state uncertainty set \(x_k \in \Omega_k\)

- **Application**:
  "Robust safety validation"
  "Reachable set Construction" for uncertain nonlinear systems
2. Risk Aware Control and Planning
2.1 Risk Bounded Trajectory Planning in Nonconvex Uncertain Environments

**Goal:** Risk Bounded Trajectory Planning in presence of perception uncertainties

**Perception Uncertainties:**
Probabilistic uncertainties in location, size, and geometry of obstacles

**Risk:** Probability of collision of robot with obstacles in presence of probabilistic uncertainties.

---

**Ordinary Map**
- **Free Region**
- **Obstacle**

**Risk Contours Map**
- Risk $\leq 0.1$
- Risk $\leq 0.3$
- Risk $\leq 0.5$
- Risk $\leq 0.7$
- Risk $\leq 0.9$
- Risk $> 0.9$
2.1 Risk Bounded Trajectory Planning in Nonconvex Uncertain Environments

- We construct a new map called “risk contours map (RCM)” that represents risk information of uncertain environment.

- We replace “risk bounded trajectory planning” with deterministic trajectory planning /path planning problem with respect to RCM.
2.2 Risk Aware Nonlinear Controller Design for Probabilistic Nonlinear Systems

- We design closed-loop controller to:
  i) drive the robot to the goal region
  ii) avoid the obstacles
in the presence of system and environment uncertainties.

- closed-loop controller in the form of "Polynomial State Feedback", i.e., \( u(x_k) = \sum_{\alpha} p_\alpha x_k^\alpha \)

- Model Predictive Control (MPC) formulation: We look for "open loop controller" \( u = [u_k, \ldots, u_{k+N}] \)

- Chance/Chance Constrained optimization formulation
2.3 Flow-Tube Based Control Of Probabilistic Nonlinear Systems

- We design a closed-loop controller (*Polynomial State Feedback*), to follow the given nominal trajectory \((x_k^*, u_k^*)\) for \(k = [0, ... N - 1]\) in presence of uncertainties.

- To cope with uncertainties, we design a closed-loop controller (*Polynomial State Feedback*) to,
  i) follow the given nominal trajectory
  ii) for safety purposes remain in the tube around the nominal trajectory, despite all uncertainties
2.3 Flow-Tube Based Control Of Probabilistic Nonlinear Systems

Obstacle Free Tube
S. Singh, M. Chen, S. L. Herbert, C. J. Tomlin, M Pavone, 2018

SpaceX

Library of tubes for real-time motion planning
A. Majumdar, R. Tedrake 2017
2.4 Chance Constrained Backward Reachable Sets For Probabilistic Nonlinear Systems

- **Backward Reachable Set**: a set of initial states $X_0$ for which target set $X_T$ is reachable in $T$ time steps under input constraints.

- **Chance Constrained Backward Reachable Set**: a set of initial states $X_0$ for which Probability of reaching the target set $X_T$ in $T$ time steps under input constraints is greater than $1 - \Delta$.

Chance Constrained Optimization Formulation
3. Risk Aware Control and Safety Verification in Presence of Gaussian Uncertainties
3. Risk Aware Control and Safety Verification in Presence of Gaussian Uncertainties:

• Dynamical Systems with **Gaussian Uncertainties:**

\[
x_{k+1} = Ax_k + Bu_k + \omega_k \\
\omega_k \sim N(0, \Sigma_k)
\]

\[
x_{k+1} = f(x_k, u_k, \omega_k) \\
\omega_k \sim N(0, \Sigma_k)
\]

• Stochastic Differential Equations (SDE)

\[
dx(t) = f(x)dt + g(x)d\omega(t) \quad \omega: \text{Brownian motion}
\]

- We will use **Gaussian distributions** to represent probability distributions of states of the system.
- We use **mean and covariance** of uncertainties.
3. Risk Aware Control and Safety Verification in Presence of Gaussian Uncertainties:

- **Dynamical Systems with Gaussian Uncertainties:**

  \[ x_{k+1} = A x_k + B u_k + \omega_k \]
  \[ \omega_k \sim N(0, \Sigma_k) \]

  \[ x_{k+1} = f(x_k, u_k, \omega_k) \]
  \[ \omega_k \sim N(0, \Sigma_k) \]

- **Stochastic Differential Equations (SDE):**

  \[ dx(t) = f(x) dt + g(x) d\omega(t) \]

  \( \omega \): Brownian motion

- **Distributionally Robust Chance Constrained Control**

  Given mean \( m^* \) and covariance \( \Sigma^* \) of uncertainties, we plan for worst-case probability distribution.

  \( Pr(m^*, \Sigma^*) \) = Family of probability distributions with mean \( m^* \) and covariance \( \Sigma^* \).

  worst-case scenario: Probability distribution \( Pr \in Pr(m^*, \Sigma^*) \) that causes highest risk in the system.

  We make sure that worst-case risk is bounded by \( 1 - \Delta \).
4. Occupation Measure and Liouville Equation
4. Occupation Measure and Liouville Equation

- We will consider nonlinear ordinary differential equation (ODE) with uncertain initial condition

\[ \dot{x}(t) = f(x(t), t) \quad x(0) \sim p_r(x_0) \]

- **Liouville’s Equation**: Linear Partial Differential Equation (PDE) that describes propagation of initial uncertainty through nonlinear ODE.

- **Occupation Measure**: distribution defined on all trajectories of the system

Distributions \( p_r(x_0), p_r(x_T), p_r(x, t) \) are connected through Liouville’s equation.
4. Occupation Measure and Liouville Equation

- We leverage Liouville’s Equation, Occupation Measures, and Moment Theory to analyze and control of nonlinear dynamical systems.

- Safety Verification

- Region of Attraction (ROA) Set Computation
  i.e., the set of all initial conditions that can be steered to the target set in an admissible way

- Optimal Control

ROA set around the origin point for Acrobot

D. Henrion, M. Korda, 2013
5. Sum-of-Squares Based Robust Control and Analyze of Uncertain Systems
5. Sum-of-Squares Based Robust Control and Analyze of Uncertain Systems

- Relies on classical definition of stability of nonlinear systems
- Lyapunov stability certificate
- SOS SDP formulation

Applications:
5.1 Lyapunov Based Stability and Region of Attraction Set,
5.2 Barrier Function Based Safety Verification,
5.3 Robust Control
Summary of Applications

1. Probabilistic Safety Verification

2. Risk Aware Control and Planning

3. Risk Aware Control and Safety Verification in Presence of Gaussian Uncertainties

4. Occupation Measure Based Control and Analyze of Nonlinear Systems

5. Sum-of-Squares Based Robust Control and Analyze of Uncertain Systems
Challenges of SDP Based Planning
We can formulate many problems in different domains as a special cases of provided optimization frameworks.

Convex formulations enable us to solve the optimization problems efficiently.

What is the Cost of Convexification?


\[\text{minimize} \quad p(x) \quad \text{subject to} \quad g_i(x) \geq 0, \quad i = 1, \ldots, n_g\]

- **Number of decision variables: \( n, (x_1, \ldots, x_n) \)**

- In the SOS based SDP we look for a polynomial \( P(x) \geq 0 \) of order \( d \).
  - **Number of decision variables: Coefficients of polynomial** \( \binom{n+d}{n} = \frac{(n+d)!}{n!d!} \)

- In the Moment based SDP, we look for a moments of probability distribution \( pr(x) \) up to order \( d \).
  - **Number of decision variables: Moments up to order \( d \)** \( \binom{n+d}{n} = \frac{(n+d)!}{n!d!} \)

- Convexification increases the space of decision variables
In the absence of problem structure, sum of squares problems are currently limited, roughly speaking, to a several thousands variables (variables in SDP).

How to address large scale problems?
1) Modified SOS optimization that results in
   i) smaller SDP’s or ii) other types of convex constraints like LP.
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   i) smaller SDP’s or ii) other types of convex constraints like LP.

2) Taking advantage of structure of the problem like sparsity

This results in following techniques:
1) Spars Sum-of-Squares Optimization (SSOS)
2) Bounded Degree Sum-of-Squares Optimization (BSOS, SBSOS)
3) (Scaled) Diagonally Dominant Sum-of-Squares Optimization (DSOS, SDSOS)
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   This results in following techniques:
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Example:

SOS based SDP problem that takes 1526.5 (s) =>
   DSOS runtime: 9.67 (s)
   SDSOS runtime: 25.9 (s)


SOS based SDP problem that takes 262.08 (s) =>
   SSOS runtime: 0.76 (s)
   DSOS runtime: 2.89 (s)
   SDSOS runtime: 5 (s)

Main Benefit: They can scale to problems where SOS programming ceases to run due to memory/computation constraints.
3) Reformulating original optimization problem to reduce the size of the problem
3) Reformulating original optimization problem to reduce the size of the problem

**Example:**

- Instead of solving a large chance optimization, we solve sequence of smaller chance optimization.
**Example:** flow-tube based control
3) Reformulating original optimization problem to reduce the size of the problem

**Example:**

- Instead of solving a large chance optimization, we solve sequence of smaller chance optimization.  
  Example: flow-tube based control

- Reducing the size of uncertain parameters:  
  Example: replacing risk estimation problem involving $n$ uncertain parameters (multivariate SOS) with univariate risk estimation problem (univariate SOS)
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  **Example:** flow-tube based control

- Reducing the size of uncertain parameters:
  **Example:** replacing risk estimation problem involving $n$ uncertain parameters (multivariate SOS) with univariate risk estimation problem (univariate SOS)

- Planning in subspace:
  **Example:** instead of constructing reachable set in $n$-dimensional state space, i.e., $(x_1, \ldots, x_n)$ construct reachable set in the subspaces of $(x_i, x_{i+1}), i = 1, \ldots, n - 1$
3) Reformulating original optimization problem to reduce the size of the problem

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- Instead of solving a large chance optimization, we solve sequence of smaller chance optimization. 
  **Example:** flow-tube based control

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4) Efficient Algorithms for Large Scale SDP’s (Guest Lecture)
Topics:

- Introduction to Planning Under Uncertainty
- Approaches and Challenges
- Technical Idea and Mathematical Tools
- Applications
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**Prerequisites:** Linear Algebra (e.g., 18.06), Convex Optimization (e.g., 6.215, 6.251, 6.255), Probability Theory (e.g., 6.431), Dynamical Systems (e.g., 6.241) or permission of the instructor.

**Office Hours:** Wednesdays 4PM-6, MIT 32-277

**Q&A on Piazza:** [https://piazza.com/mit/fall2019/16s498/home](https://piazza.com/mit/fall2019/16s498/home)

**Lecture Slides:** All lectures will be posted on Tuesdays in the course website: [https://stellar.mit.edu/S/course/16/fa19/16.S498](https://stellar.mit.edu/S/course/16/fa19/16.S498)

**Code Repository:** [https://github.com/jasour/rarnop19](https://github.com/jasour/rarnop19)

**Assignments and Grading:** 50% Problem Sets, 50% Research Project

Problem sets will be posted in the course website and will be due one week later.

**Bibliography:** Variety of book and recent papers will be introduced for each lecture.
Research Project

- Apply the provided techniques to your research problems.

- Implementation of other techniques that address uncertain nonlinear problems.

- Research Projects, i.e. improving and extending the state-of-the-art

Deadlines:

November 1\textsuperscript{st}: Submit a short description of the project (max 1 page)
December 11: Final Project Presentation